Homological operations

Recall from lecture 1:

$R = \mathbb{C}[x_1 \dots x_n]$

β = braid on n strands $\rightsquigarrow$

T_β = complex of R-R bimodules

= modules over $\mathbb{C}[\underline{x}_1 \dots x_n, x_1' \dots x_n']$

What are the general properties of T_β?

① For any symmetric function f

$f(x_1 \dots x_n) = f(x_1' \dots x_n')$ on T_β

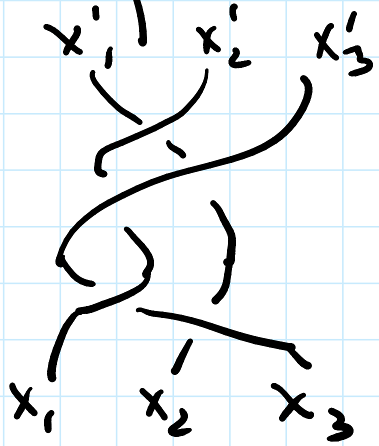
More abstractly, consider

$$B = B_{w_0} = \frac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{\left(\begin{array}{c} f(x_1 \dots x_n) = f(x_1' \dots x_n') \\ \text{for all symmetric} \\ \text{functions } f \end{array} \right)}$$

Then the action of $\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']$ on T_β factors through B.

② The action of x_i is homotopic to the action of $x'_{w(i)}$ where

is the action of $x_{w(i)}$ where
w = permutation corresponding to β



$x_1 \sim x_2'$
$x_2 \sim x_1'$
$x_3 \sim x_3'$

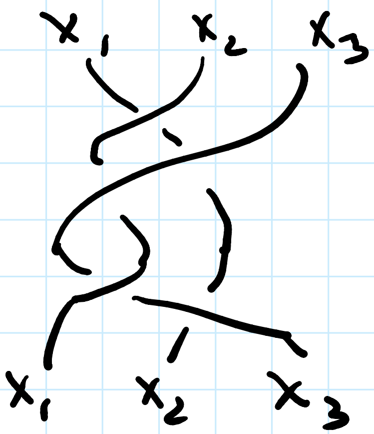
After closure: $x_1 \sim \underbrace{x_2' = x_2}$, $x_2 \sim \underbrace{x_1' = x_1}$, $x_3 \sim \underbrace{x_3' = x_3}$

Important corollary L = closure of β = link with r components
$\Rightarrow$ HHH(L) is naturally a module over a polynomial ring, one variable per component of L.

[Note: components of $L \longleftrightarrow$ cycles in the permutation w corresponding to β]

③ Can study homotopies between x_i and $x'_{w(i)}$ more closely.

x_1 x_2 x_3



$d(\xi_1) = x_1 - x_2'$

$d(\xi_2) = x_2 - x_1'$

$d(\xi_3) = x_3 - x_3'$

<u>After closure</u>: $d(\xi_1) = x_1 - x_2$, $d(\xi_2) = x_2 - x_1$

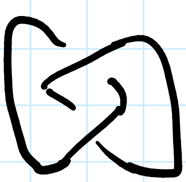
$d(\xi_1 + \xi_2) = 0$! $\qquad d(\xi_3) = x_3 - x_3 = 0$. !

$\xi_1 + \xi_2, \ \xi_3 \longrightarrow$ nontrivial homological generations, "monodromy"

Can use ξ_i to deform the differential:

$$D = d + \sum \xi_i y_i \ \circlearrowright \ T_\beta \otimes \mathbb{C}[y_1 \ldots y_n]$$

y_i = formal variables

Ex  $\rightsquigarrow$
$$\begin{tikzcd} \left[R \arrow[r, "0"] & R \arrow[r, "x_1 - x_2"] \arrow[l, bend left, "\xi"] & R \right] \end{tikzcd}$$ (Lecture 1)

Deformed complex:
$$\begin{tikzcd} \left[R[y] \arrow[r, "0"] & R[y] \arrow[r, "x_1 - x_2"] \arrow[l, bend left, "y_1 - y_2"] & R[y] \right] \end{tikzcd}$$
$z \qquad\qquad\qquad\qquad w$

Homology $= \dfrac{\mathbb{C}[x_1, x_2, y_1, y_2]\langle z, w\rangle}{}$

Homology $= \dfrac{\mathbb{C}[x_1, x_2, y_1, y_2]\langle z, w\rangle}{(z(y_1-y_2) = w(x_1-x_2))}$
HY

<u>Thm</u> (G., Hogancamp) $T(n,kn)$ = torus link with n components

(a) $HY(T(n,kn)) = J^k$ as triply graded $R[y]$-modules.

deformed homology

where $J = \bigcap_{i \neq j} (x_i - x_j, y_i - y_j, \theta_i - \theta_j)$

$= \text{ideal in } \mathbb{C}[\underbrace{x_1 \dots x_n, y_1 \dots y_n}_{\text{even}}, \underbrace{\theta_1 \dots \theta_n}_{\text{odd}}]$.

In HH^0 we get just

$\bigcap_{i \neq j} (x_i - x_j, y_i - y_j) \subset \mathbb{C}[x_1 \dots x_n, y_1 \dots y_n]$
$\quad\quad \shortparallel$
ideal defining the diagonal in $(\mathbb{C}^2)^n$

(b) $HHH(T(n,kn)) = J^k \big/ (y) J^k$

and J^k is free over $\mathbb{C}[y_1 \dots y_n]$

<u>Rmk</u>: Much easier to describe the deformed homology first!

(4) A = resolution of R over $B = \dfrac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{\mathbb{C}[x_1 \dots x_n]^{S_n} \dots}$

(4) $\mathcal{A}$ = resolution of R over $B = \dfrac{\mathbb{C}[x_1 \dots x_n, x_1' \dots x_n']}{\left(\begin{array}{l} f(x_1 \dots x_n) = \\ f(x_1' \dots x_n') \\ f \text{ symmetric} \end{array} \right)}$

Concretely:

$\mathcal{A} = B[\xi_1 \dots \xi_n, u_1 \dots u_n]$

$d(\xi_i) = x_i - x_i'$ ⟵ we saw already

$d(u_k) = \sum_{i=1}^{n} h_{k-1}(x_i, x_i')\,\xi_i$

complete symmetric function

Exercise: $d^2 = 0$ & this is indeed a resolution

Thm (G., Hogancamp, Mellit)

The dg algebra $\mathcal{A}$ acts on the Rouquier complex T_β for any β.

Pf:

- $T_i = [B_i \to R]$ $\quad T_i^{-1} = [R \to B_i]$

  Construct ξ_i explicitly, $u_k = 0$

- There's a coproduct

$$\Delta : \mathcal{A} \longrightarrow \overset{x_i}{\mathcal{A}} \overset{x_i'}{\underset{R}{\otimes}} \mathcal{A} \quad x_i''$$

$\Delta(x_i) = x_i \otimes 1 \qquad \Delta(x_i') = 1 \otimes x_i' = x_i''$

$\Delta(x_i) = x_i \otimes 1 \qquad \Delta(x_i) = 1 \otimes x_i = x_i$

$\Delta(\xi_i) = \xi_i \otimes 1 + 1 \otimes \xi_i$

$$\Delta(u_k) = u_k \otimes 1 + 1 \otimes u_k + \sum_{i=1}^{n} h_{k-2}(x_i, x_i', x_i'')\, \xi_i \otimes \xi_i$$

If M, N are $\mathcal{A}$-modules then $M \otimes_R N$ is via coproduct.

(5) To sum up: there is an action of interesting operators ξ_i, u_k on T_β but these are not closed!

<u>Solution</u>: deformed homology

$$D = d + \sum \xi_i y_i$$

$$F_k = u_k + \sum h_{k-1}(x_i, x_i') \frac{\partial}{\partial y_i}$$

<u>Thm</u> [GHM] (a) $[D, F_k] = 0$, $[F_k, F_\ell] = 0$

$[F_k, x_i] = 0$, $[F_k, y_i] = h_{k-1}(x_i, x_i')$

F_k = "tautological classes" $\quad k x_i^{k-1}$ after closure.

(b) F_2 satisfies "hard Lefshetz" condition and lifts to an action of condition and lifts to an action of sl_2 on $HY(L)$ for any link L

Cor $HY(L)$ is symmetric!

(Conjectured by Gukov, Dunfield, Rasmussen).

(6) Geometric analogue:

tautological classes on braid/character varieties.

$F_k \rightsquigarrow$ algebraic k-form on $X(\beta)$

Bott, Shulman, ...: $G = GL_n$

Q = sym. function of degree r

$\to \Phi_0(Q) \in H^{2r}(BG)$

$\to \Phi_1(Q) \in H^{2r-1}(G) \quad d\Phi_1(Q) = 0$

$\to \Phi_2(Q) \in H^{2r-2}(G \times G)$

$$d\Phi_2(Q) = \pi_1^* \Phi_1(Q) - m^* \Phi_1(Q) + \pi_2^* \Phi_1(Q)$$

$m : G \times G \to G$ multiplication

Ex: $Q = \sum x_i^2$

$$\Phi_2(Q) = \mathrm{Tr}(f^{-1} df \wedge dg\, g^{-1})$$

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Given
$$\begin{tikzcd} X \arrow[r, "f"] & G \\ Y \arrow[ur, "g"'] & \end{tikzcd} \qquad \begin{aligned} f^*\Phi_1(Q) &= d\omega_X \\ g^*\Phi_1(Q) &= d\omega_Y \end{aligned}$$

Can glue: $\omega_{X\times Y} = \omega_X + \omega_Y + \Phi_2(Q)$

form on $X \times Y$.

Similar to coproduct on $\mathcal{A}$.

Thm (Mellit) One can use this to construct a 2-form on $X(\beta)$ which satisfies "curious hard Lefschetz" wrt weight filtration.